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MANIN TRIPLES FOR LIE BIALGEBROIDS

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Abstract

In his study of Dirac structures, a notion which includes both Poisson structures and closed 2-forms, T. Courant introduced a bracket on the direct sum of vector fields and 1-forms. This bracket does not satisfy the Jacobi identity except on certain subspaces. In this paper we systematize the properties of this bracket in the definition of a **Courant algebroid**. This structure on a vector bundle $E \to M$, consists of an antisymmetric bracket on the sections of E whose "Jacobi anomaly" has an explicit expression in terms of a bundle map $E \to TM$ and a field of symmetric bilinear forms on E. When M is a point, the definition reduces to that of a Lie algebra carrying an invariant nondegenerate symmetric bilinear form.

For any Lie bialgebroid (A, A^*) over M (a notion defined by Mackenzie and Xu), there is a natural Courant algebroid structure on $A \oplus A^*$ which is the Drinfel'd double of a Lie bialgebra when M is a point. Conversely, if A and A^* are complementary isotropic subbundles of a Courant algebroid E, closed under the bracket (such a bundle, with dimension half that of E, is called a **Dirac structure**), there is a natural Lie bialgebroid structure on (A, A^*) whose double is isomorphic to E. The theory of Manin triples is thereby extended from Lie algebras to Lie algebroids.

Our work gives a new approach to bihamiltonian structures and a new way of combining two Poisson structures to obtain a third one. We also take some tentative steps toward generalizing Drinfel'd's theory of Poisson homogeneous spaces from groups to groupoids.

1. Introduction

The aim of this paper is to solve, in a unified way, several mysteries which have arisen over the past few years in connection with generalizations of the notion of Lie algebra in differential geometry.

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T. Courant [4] introduced the following antisymmetric bracket operation on the sections of $TP \oplus T^*P$ over a manifold P:

$$[X_1+\xi_1, X_2+\xi_2] = [X_1, X_2] + (L_{X_1}\xi_2 - L_{X_2}\xi_1 + d(\tfrac{1}{2}(\xi_1(X_2) - \xi_2(X_1))).$$

Were it not for the last term, this would be the bracket for the semidirect product of the Lie algebra $\mathcal{X}(P)$ of vector fields with vector space $\Omega^1(P)$ of 1-forms via the Lie derivative representation of $\mathcal{X}(P)$ on $\Omega^1(P)$. The last term, which was essential for Courant's work (about which more will be said later) causes the Jacobi identity to fail. Nevertheless, for subbundles $E \subseteq TP \oplus T^*P$ which are maximally isotropic for the bilinear form $(X_1+\xi_1, X_2+\xi_2)_+ = \frac{1}{2}(\xi_1(X_2)+\xi_2(X_1))$, closure of $\Gamma(E)$ under the Courant bracket implies that the Jacobi identity *does* hold on $\Gamma(E)$, because of the maximal isotropic condition on E. These subbundles are called *Dirac structures* on P; the notion is a simultaneous generalization of that of Poisson structure (when E is the graph of a map $\tilde{\pi} : T^*P \to TP$) and that of closed 2-form (when E is the graph of a map $\tilde{\omega} : TP \to T^*P$).

Problem 1. Since the Jacobi identity is satisfied on certain subspaces where $(\ ,\ )_+$ vanishes, find a formula for the *Jacobi anomaly*[1]

$$[[e_1, e_2], e_3] + c.p.$$

in terms of $(\ ,\ )_+$.

The vector space $\chi(P) \oplus \Omega^1(P)$ on which the Courant bracket is defined is also a module over $C^\infty(P)$. Projection on the first factor defines a map ρ from $\chi(P) \oplus \Omega^1(P)$ to derivations of $C^\infty(P)$. If one checks the Leibniz identity which enters in the definition of a *Lie algebroid* [20],

$$[e_1, fe_2] = f[e_1, e_2] + (\rho(e_1)f)e_2\,.$$

It turns out that this is not satisfied in general, but that it is satisfied for Dirac structures. This suggests:

Problem 2. Express the *Leibniz anomaly* $[e_1, fe_2] - f[e_1, e_2] - (\rho(e_1)f)e_2$ in terms of $(\ ,\ )_+$.

When one is given an inner product on a Lie algebra, it is natural to ask whether it is invariant under the adjoint representation. Here again, a calculation turns up an *invariance anomaly.*

[1] "$+c.p.$" below (and henceforth) will denote "plus the other two terms obtained by circular permutations of $(1, 2, 3)$."

We solve problems 1 and 2 in the paper, finding an expression for the invariance anomaly as well. The formulas obtained are so attractive as to suggest:

Problem 3. Generalize the Courant bracket by writing down a set of axioms for a skew-symmetric bracket $\mathcal{E} \times \mathcal{E} \to \mathcal{E}$, a linear map $\mathcal{E} \to \mathrm{Der}(C^\infty(M))$, and a symmetric inner product $\mathcal{E} \times \mathcal{E} \to C^\infty(M)$ on the space $\mathcal{E}$ of sections of a vector bundle over M, and find other interesting examples of the structure thus defined.

Our solution of Problem 3 begins with the definition of a structure which we call a *Courant algebroid*[2]. Among the examples of Courant algebroids which we find are the direct sum of any Lie bialgebroid [22] and its dual, with the bracket given by a symmetrized version of Courant's original definition. This structure thus gives an answer as well to:

Problem 4. What kind of object is the double of a Lie bialgebroid?

Furthermore, within each Courant algebroid, one can consider the maximal isotropic subbundles closed under bracket. These more general Dirac structures are new Lie algebroids (and sometimes Lie bialgebroids). Constructions in this framework applied to the Lie bialgebroid of a Poisson manifold [22] lead to new ways of building Poisson structures and shed new light on the theory of Poisson-Nijenhuis structures used to explicate the hamiltonian theory of completely integrable systems [13]. In particular, we find a composition law for certain pairs of (possibly degenerate) Poisson structures which generalizes the addition of symplectic structures: namely, if $U : T^*P \to TP$ and $V : T^*P \to TP$ define Poisson structures such that $U + V$ is invertible, then $U(U+V)^{-1}V$ again defines a Poisson structure.

When the base manifold P is a point, a Lie algebroid is just a Lie algebra. A Courant algebroid over a point turns out to be nothing but a Lie algebra equipped with a nondegenerate ad-invariant symmetric 2-form (sometimes called an orthogonal structure [25]). (The formulas for the anomalies all involve derivatives, so they vanish when P is a point.) Such algebras and their maximal isotropic subalgebras are the ingredients of the theory of Lie bialgebras and Manin triples [6]. In fact, just as a complementary pair of isotropic subalgebras in a Lie algebra with orthogonal structure determines a Lie bialgebra, so a complementary pair of Dirac structures in a Courant algebroid determines a Lie

[2]We apologize to our French colleagues for possible confusion with the nearly homonymous and somewhat less synonymous term, "algèbre de courants".

bialgebroid. It is this fact, which exhibits our theory as a generalization of the theory of Manin triples, which is responsible for the application to Poisson-Nijenhuis pairs mentioned above.

We mentioned earlier that the notion of Dirac structures was invented in order to treat in the same framework Poisson structures, which satisfy the equation $[\pi, \pi] = 0$, and closed 2-forms, which satisfy $d\omega = 0$. One could look for a more direct connection between these equations.

Problem 5. What is the relation between the equations $[\pi, \pi] = 0$ and $d\omega = 0$?

Our solution to Problem 5 is very simple. In a Courant algebroid of the form $A \oplus A^*$, the double of a Lie bialgebroid, the equation which a skew-symmetric operator $\tilde{I} : A \to A^*$ must satisfy in order for its graph to be a Dirac structure turns out to be the Maurer-Cartan equation $dI + \frac{1}{2}[I, I] = 0$ for the corresponding bilinear form $I \in \Gamma(\wedge^2 A^*)$. The structure of the original Courant algebroid $TM \oplus TM^*$ (also viewed dually as $T^*M \oplus TM$) is sufficiently degenerate that one of the terms in the Maurer-Cartan equation drops out in each of the two cases.

The next problem arises from Drinfeld's study [8] of Poisson homogeneous spaces for Poisson Lie groups. He shows in that paper that the Poisson manifolds on which a Poisson Lie group G acts transitively are essentially (that is, if one deals with local rather than global objects, as did Lie in the old days) in 1-1 correspondence with Dirac subspaces of the double of the associated Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$. It is natural, then, to look for some kind of homogeneous space associated to a Dirac subbundle in the double of a Lie bialgebroid.

The object of which a Lie bialgebroid $E \longrightarrow P$ is the infinitesimal limit is a Poisson groupoid, i.e. a Poisson manifold Γ carrying the structure of a groupoid with base P, for which the graph of multiplication $\{(k, g, h)|k = gh\}$ is a coisotropic submanifold of $\Gamma \times \overline{\Gamma} \times \overline{\Gamma}$. ($\overline{\Gamma}$ is Γ with the opposite Poisson structure. See [22] [23] and [33].) Unlike in the case of groups, a Poisson groupoid corresponding to a given Lie bialgebroid may exist only locally.

Problem 6. Define a notion of Poisson homogeneous space for a Poisson groupoid. Show that Dirac structures in the double of a Lie bialgebroid (A, A^*) correspond to (local) Poisson homogeneous spaces for the (local) Poisson groupoid Γ associated to (A, A^*).

Our solution to Problem 6 will be contained in a sequel to this paper [17]. Even if we work locally, it is somewhat complicated, since the "homogeneous spaces" for groupoids, which are already hard to define in general (see [3]), in this case can involve the quotient spaces of manifolds by arbitrary foliations.

To give a flavor of our results, we mention here one example. For the standard Lie bialgebroid (TM, T^*M), the associated Poisson groupoid is the pair groupoid $M \times M$ with the zero Poisson structure. A Dirac structure transverse to T^*M is the graph of a closed 2-form ω on M. The corresponding Poisson homogeneous space for $M \times M$ is $M \times (M/\mathcal{F})$, where the factor M has the zero Poisson structure, and the factor $M/\mathcal{F}$ is the (symplectic) Poisson manifold obtained from reduction of M by the characteristic foliation $\mathcal{F}$ of ω. (Of course, the leaf space $M/\mathcal{F}$ might not be a manifold in any nice sense.) Dually, our Dirac structure also defines a Poisson homogeneous space for the Poisson groupoid of the Lie bialgebroid (T^*M, TM), which is T^*M with the operation of addition in fibres and the Poisson structure given by the canonical 2-form. The homogeneous space is again T^*M, with the Poisson structure coming from the sum of the canonical 2-form and the pullback of ω by the projection $T^*M \longrightarrow M$.

We turn now to some problems which remain unsolved.

The only examples of Courant algebroids which we have given are the doubles of Lie bialgebroids, i.e. those admitting a direct sum decomposition into Dirac subbundles. For Courant algebroids over a point, there are many examples which are not of this type, even when the symmetric form has signature zero, which is necessary for such a decomposition. For instance, we may take the direct sum of two Lie algebras of dimension k with invariant bilinear forms, one positive definite and one negative definite. Any isotropic subalgebra of dimension k must be the graph of an orthogonal isomorphism from one algebra to the other. Such an isomorphism may not exist. Even if it does, it might be the case that the graphs of any two such isomorphisms must have a line in common. (For instance, take two copies of $\mathfrak{su}(2)$ and use the fact that every rotation of $\mathbb{R}^3$ has an axis.) These examples and a further study of Manin triples from the point of view of Lie algebras with orthogonal structure may be found in [26].

Open Problem 1. Find interesting examples of Courant algebroids which are not doubles of Lie bialgebroids, including examples which admit one Dirac subbundle, but not a pair of transverse ones. Are there Courant algebroids which are not closely related to finite dimensional Lie algebras, for which the bilinear form is positive definite?

In his study of quantum groups and the Knizhnik-Zamolodchikov equation, Drinfeld [7] introduced quasi-Hopf algebras, in which the axiom of coassociativity is weakened, and their classical limits, the Lie quasi-bialgebras. The latter notion was studied in depth by Kosmann-Schwarzbach [11] (see also [2]), who defined various structures involving a pair of spaces in duality carrying skew symmetric brackets whose Jacobi anomalies appear as coboundaries of other objects. Her structures are not subsumed by ours, though, since our expression for the Jacobi anomaly is zero when the base manifold is a point. Jacobi anomalies as coboundaries also appear in the theory of "strongly homotopy Lie algebras" [14] and in recent work of Ginzburg [9]. The relation of these studies to Courant algebroids is the subject of work in progress with Dmitry Roytenberg.

Open Problem 2. Define an interesting type of structure which includes both the Courant algebroids and the Lie quasi-bialgebras as special cases.

At the very beginning of our study, we found that if the bracket on a Courant algebroid is modified by the addition of a symmetric term, many of the anomalies for the resulting asymmetric bracket become zero. This resembles the "twisting" phenomenon of Drinfeld [7].

Open Problem 3. What is the geometric meaning of such asymmetric brackets, satisfying most of the axioms of a Lie algebroid?

The next problem is somewhat vague. The Maurer-Cartan equation $d\alpha + \frac{1}{2}[\alpha, \alpha]$ appears as an integrability condition in the theory of connections and plays an essential role in modern deformation theory. (See [28] and various original sources cited therein.)

Open Problem 4. Find geometric or deformation-theoretic interpretations of the Maurer-Cartan equation for Dirac structures.

Lie algebras, Lie algebroids and (the doubles of) Lie bialgebras are the infinitesimal objects corresponding to Lie groups, Lie groupoids, and (the doubles of) Poisson Lie groups respectively. Moreover, Kosmann-Schwarzbach [11] has studied the global objects corresponding to Lie quasi-bialgebras, and Bangoura [1] has recently identified the dual objects. Yet the following problem is unsolved, even for $TM \oplus T^*M$.

Open Problem 5. What is the global, groupoid-like object corresponding to a Courant algebroid? In particular, what is the double of a Poisson groupoid?

A solution to Open Problem 4 might come from a solution to the next problem. When one passes from an object such as a Lie bialgebra or even a Lie quasi-bialgebra to its double, the resulting object is frequently "nicer" in the sense that some of the anomalies possessed by the original object now vanish.

Open Problem 6. What is the double of a Courant algebroid?

Finally, we would like to remark that many of the constructions in this paper can be carried out at a more abstract level, either replacing the sections of a vector bundle E by a more general module over $C^\infty(P)$, as in [10], or in the context of local functionals on mapping spaces as in [5] by Dorfman.

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2. Double of Lie bialgebroids

Definition 2.1. A *Courant algebroid* is a vector bundle $E \longrightarrow P$ equipped with a nondegenerate symmetric bilinear form $(\cdot,\cdot)$ on the bundle, a skew-symmetric bracket $[\cdot,\cdot]$ on $\Gamma(E)$ and a bundle map $\rho : E \longrightarrow TP$ such that the following properties are satisfied:

(i) For any $e_1, e_2, e_3 \in \Gamma(E)$, $[[e_1, e_2], e_3] + c.p. = \mathcal{D}T(e_1, e_2, e_3)$;

(ii) for any $e_1, e_2 \in \Gamma(E)$, $\rho[e_1, e_2] = [\rho e_1, \rho e_2]$;

(iii) for any $e_1, e_2 \in \Gamma(E)$ and $f \in C^\infty(P)$,

$$[e_1, fe_2] = f[e_1, e_2] + (\rho(e_1)f)e_2 - (e_1, e_2)\mathcal{D}f;$$

(iv) $\rho{\circ}\mathcal{D} = 0$, i.e., for any $f, g \in C^\infty(P)$, $(\mathcal{D}f, \mathcal{D}g) = 0$;

(v) for any $e, h_1, h_2 \in \Gamma(E)$,

$$\rho(e)(h_1, h_2) = ([e, h_1] + \mathcal{D}(e, h_1), h_2) + (h_1, [e, h_2] + \mathcal{D}(e, h_2)),$$

where $T(e_1, e_2, e_3)$ is the function on the base P defined by:

$$T(e_1, e_2, e_3) = \frac{1}{3}([e_1, e_2], e_3) + c.p., \tag{1}$$

and $\mathcal{D} : C^\infty(P) \longrightarrow \Gamma(E)$ is the map defined[3] by $\mathcal{D} = \frac{1}{2}\beta^{-1}\rho^* d_0$, β being the isomorphism between E and E^* given by the bilinear form. In other words,

$$(\mathcal{D}f, e) = \frac{1}{2}\rho(e)f. \tag{2}$$

Remark. Introduce a twisted bracket (not antisymmetric!) on $\Gamma(E)$ by

$$[e, h]\tilde{} = [e, h] + \mathcal{D}(e, h).$$

Then (iii) is equivalent to

$$[e_1, fe_2]\tilde{} = f[e_1, e_2]\tilde{} + (\rho(e_1)f)e_2; \tag{3}$$

(v) is equivalent to

$$\rho(e)(h_1, h_2) = ([e, h_1]\tilde{}, h_2) + (h_1, [e, h_2]\tilde{}); \tag{4}$$

and (ii) and (iv) can be combined into a single equation:

$$\rho[e_1, e_2]\tilde{} = [\rho e_1, \rho e_2]. \tag{5}$$

It would be nice to interpret equation (i) in terms of this twisted bracket. The geometric meaning of this twisted bracket remains a mystery to us.

Definition 2.2. Let E be a Courant algebroid. A subbundle L of E is called *isotropic* if it is isotropic under the symmetric bilinear form $(\cdot, \cdot)$. It is called *integrable* if $\Gamma(L)$ is closed under the bracket $[\cdot, \cdot]$. A *Dirac structure,* or *Dirac subbundle,* is a subbundle L which is maximally isotropic and integrable.

The following proposition follows immediately from the definition.

Proposition 2.3. *Suppose that L is an integrable isotropic subbundle of a Courant algebroid $(E, \rho, [\cdot, \cdot], (\cdot, \cdot))$. Then $(L, \rho|_L, [\cdot, \cdot])$ is a Lie algebroid.*

[3]In this paper, d_0 denotes the usual differential from functions to 1-forms, while d will denote the differential from functions to sections of the dual of a Lie algebroid.

Suppose now that both A and A^* are Lie algebroids over the base manifold P, with anchors a and a_* respectively. Let E denote their vector bundle direct sum: $E = A \oplus A^*$. On E, there exist two natural nondegenerate bilinear forms, one symmetric and another antisymmetric, which are defined as follows:

$$(X_1 + \xi_1, X_2 + \xi_2)_{\pm} = \frac{1}{2}(\langle \xi_1, X_2 \rangle \pm \langle \xi_2, X_1 \rangle). \tag{6}$$

On $\Gamma(E)$, we introduce a bracket by

$$\begin{aligned}[e_1, e_2] =& ([X_1, X_2] + L_{\xi_1} X_2 - L_{\xi_2} X_1 - d_*(e_1, e_2)_-) \\ &+ ([\xi_1, \xi_2] + L_{X_1} \xi_2 - L_{X_2} \xi_1 + d(e_1, e_2)_-),\end{aligned} \tag{7}$$

where $e_1 = X_1 + \xi_1$ and $e_2 = X_2 + \xi_2$.

Finally, we let $\rho : E \longrightarrow TP$ be the bundle map defined by $\rho = a + a_*$. That is,

$$\rho(X + \xi) = a(X) + a_*(\xi), \quad \forall X \in \Gamma(A) \text{ and } \xi \in \Gamma(A^*). \tag{8}$$

It is easy to see that in this case the operator $\mathcal{D}$ as defined by Equation (2) is given by

$$\mathcal{D} = d_* + d,$$

where $d_* : C^\infty(P) \longrightarrow \Gamma(A)$ and $d : C^\infty(P) \longrightarrow \Gamma(A^*)$ are the usual differential operators associated to Lie algebroids [22].

When (A, A^*) is a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$, the bracket above reduces to the famous Lie bracket of Manin on the double $\mathfrak{g} \oplus \mathfrak{g}^*$. On the other hand, if A is the tangent bundle Lie algebroid TM and $A^* = T^*M$ with zero bracket, then Equation (7) takes the form:

$$[X_1 + \xi_1, X_2 + \xi_2] = [X_1, X_2] + \{L_{X_1} \xi_2 - L_{X_2} \xi_1 + d(e_1, e_2)_-\}.$$

This is the bracket first introduced by Courant [4], and then generalized to the context of the formal variational calculus by Dorfman [5].

Our work in this paper is largely motivated by an attempt to unify the two examples above, based on the observation that Courant's bracket appears to be some kind of "double." In order to generalize Manin's construction to Lie algebroids, it is necessary to have a compatibility condition between Lie algebroid structures on a vector bundle and its dual. Such a condition, providing a definition of *Lie bialgebroid*, was given in [22]. We quote here an equivalent formulation from [12].

Definition 2.4. A *Lie bialgebroid* is a dual pair (A, A^*) of vector bundles equipped with Lie algebroid structures such that the differential d_* on $\Gamma(\wedge^* A)$ coming from the structure on A^* is a derivation of the Schouten-type bracket on $\Gamma(\wedge^* A)$ obtained by extension of the structure on A.

The following two main theorems of this paper show that we have indeed found the proper version of the theory of Manin triples for the Lie algebroid case.

Theorem 2.5. *If (A, A^*) is a Lie bialgebroid, then $E = A \oplus A^*$ together with $([\cdot,\cdot], \rho, (\cdot,\cdot)_+)$ is a Courant algebroid.*

Conversely, we have

Theorem 2.6. *In a Courant algebroid $(E, \rho, [\cdot,\cdot], (\cdot,\cdot))$, suppose that L_1 and L_2 are Dirac subbundles transversal to each other, i.e., $E = L_1 \oplus L_2$. Then, (L_1, L_2) is a Lie bialgebroid, where L_2 is considered as the dual bundle of L_1 under the pairing $2(\cdot,\cdot)$.*

An immediate consequence of the theorems above is the following duality property of Lie bialgebroids, which was first proved in [22] and then by Kosmann-Schwarzbach [12] using a simpler method.

Corollary 2.7. *If (A, A^*) is a Lie bialgebroid, so is (A^*, A).*

3. Jacobi anomaly

In this section, we begin the computations leading to the proofs of our main theorems. Throughout this section, we assume that A is a Lie algebroid with anchor a and that its dual A^* is also equipped with a Lie algebroid structure with anchor a_*. However, we shall *not* assume any compatibility conditions between these two algebroid structures.

For simplicity, for any $e_i = X_i + \xi_i \in \Gamma(E)$, $i = 1, 2, 3$, we let

$$J(e_1, e_2, e_3) = [[e_1, e_2], e_3] + c.p.$$

The main theorem of this section is the following.

Theorem 3.1. *Assume that both (A, a) and (A^*, a_*) are Lie alge-*

broids. Then, for $e_i = X_i + \xi_i \in \Gamma(E), i = 1, 2, 3$, we have

$$J(e_1, e_2, e_3) = \mathcal{D}T(e_1, e_2, e_3) - (J_1 + J_2 + c.p), \tag{9}$$

where

$$\begin{aligned} J_1 =& i_{X_3}(d[\xi_1, \xi_2] - L_{\xi_1} d\xi_2 + L_{\xi_2} d\xi_1) + i_{\xi_3}(d_*[X_1, X_2] \\ & - L_{X_1} d_* X_2 + L_{X_2} d_* X_1), \end{aligned}$$

and

$$J_2 = L_{d_*(e_1,e_2)_-}\xi_3 + [d(e_1, e_2)_-,\ \xi_3] + L_{d(e_1,e_2)_-} X_3 + [d_*(e_1, e_2)_-,\ X_3].$$

We need a series of lemmas before proving this theorem.

Lemma 3.2. *For $e_i = X_i + \xi_i \in \Gamma(E), i = 1, 2, 3$, T is skew-symmetric, and*

$$\begin{aligned} T(e_1, e_2, e_3) =& \frac{1}{2}\Big\{\langle [X_1, X_2], \xi_3\rangle + \langle [\xi_1, \xi_2], X_3\rangle \\ & \quad + a(X_3)(e_1, e_2)_- - a_*(\xi_3)(e_1, e_2)_-\Big\} \\ & + c.p. \end{aligned} \tag{10}$$

Proof. The first assertion is obvious from the definition of T. For the second one, we first have

$$\begin{aligned} &([e_1, e_2], e_3)_+ \\ &= \frac{1}{2}\Big\{\langle [X_1, X_2], \xi_3\rangle + \langle L_{\xi_1} X_2, \xi_3\rangle - \langle L_{\xi_2} X_1, \xi_3\rangle - a_*(\xi_3)(e_1, e_2)_- \\ &\qquad + \langle [\xi_1, \xi_2], X_3\rangle + \langle L_{X_1}\xi_2, X_3\rangle - \langle L_{X_2}\xi_1, X_3\rangle + a(X_3)(e_1, e_2)_-\Big\} \\ &= \frac{1}{2}\Big\{[\langle [X_1, X_2], \xi_3\rangle + \langle [\xi_1, \xi_2], X_3\rangle] + c.p.\Big\} \\ &\quad + \frac{1}{2}\Big\{a_*(\xi_1)\langle X_2, \xi_3\rangle - a_*(\xi_2)\langle X_1, \xi_3\rangle - a_*(\xi_3)(e_1, e_2)_- \\ &\qquad + a(X_1)\langle \xi_2, X_3\rangle - a(X_2)\langle \xi_1, X_3\rangle + a(X_3)(e_1, e_2)_-\Big\} \\ &= \frac{1}{2}\Big[\Big\{\langle [X_1, X_2], \xi_3\rangle + \langle [\xi_1, \xi_2], X_3\rangle + a(X_3)(e_1, e_2)_- \\ &\qquad - a_*(\xi_3)(e_1, e_2)_-\Big\} + c.p.\Big] \\ &\quad + \frac{1}{2}\rho(e_1)(e_2, e_3)_+ - \frac{1}{2}\rho(e_2)(e_3, e_1)_+. \end{aligned}$$

Therefore, by taking the sum of its cyclic permutations, one obtains

$$\begin{aligned} T(e_1,e_2,e_3) &= \frac{1}{3}([e_1,e_2],e_3)_+ + c.p. \\ &= \frac{1}{2}\Big\{\langle [X_1,X_2],\xi_3\rangle + \langle [\xi_1,\xi_2],X_3\rangle + a(X_3)(e_1,e_2)_- \\ &\qquad -a_*(\xi_3)(e_1,e_2)_-\Big\} + c.p. \end{aligned}$$

q.e.d.

As a by-product, we obtain the following identity by substituting Equation (10) into the last step of the computation of $([e_1,e_2],e_3)_+$ in the proof above. This formula will be useful later.

$$\begin{aligned} ([e_1,e_2],e_3)_+ =& T(e_1,e_2,e_3) + \frac{1}{2}\rho(e_1)(e_2,e_3)_+ \\ & - \frac{1}{2}\rho(e_2)(e_3,e_1)_+. \end{aligned} \tag{11}$$

Lemma 3.3.

$$\begin{aligned} i_X L_\xi d\eta =& [\xi, L_X\eta] - L_{L_\xi X}\eta + [d\langle \eta, X\rangle,\ \xi] \\ & + d(a_*(\xi)\langle \eta, X\rangle) - d\langle [\xi,\eta], X\rangle. \end{aligned} \tag{12}$$

Proof. For any $Y \in \Gamma(A)$,

$$\begin{aligned} \langle i_X L_\xi d\eta, Y\rangle &= (L_\xi d\eta)(X,Y) \\ &= a_*(\xi)[d\eta(X,Y)] - d\eta(L_\xi X, Y) - d\eta(X, L_\xi Y) \\ &= a_*(\xi)a(X)\langle \eta, Y\rangle - a_*(\xi)a(Y)\langle \eta, X\rangle \\ &\quad -a_*(\xi)\langle \eta, [X,Y]\rangle - a(L_\xi X)\langle \eta, Y\rangle \\ &\quad +a(Y)\langle \eta, L_\xi X\rangle + \langle \eta, [L_\xi X, Y]\rangle \\ &\quad -a(X)\langle \eta, L_\xi Y\rangle + a(L_\xi Y)\langle \eta, X\rangle + \langle \eta, [X, L_\xi Y]\rangle \\ &= a_*(\xi)\langle L_X\eta, Y\rangle - a_*(\xi)a(Y)\langle \eta, X\rangle + a(Y)a_*(\xi)\langle \eta, X\rangle \\ &\quad -a(Y)\langle [\xi,\eta], X\rangle - \langle L_{L_\xi X}\eta, Y\rangle \\ &\quad -\langle L_X\eta, L_\xi Y\rangle + \langle L_\xi Y, d\langle \eta, X\rangle\rangle \\ &= \langle [\xi, L_X\eta], Y\rangle + \langle [d\langle \eta, X\rangle, \xi], Y\rangle \\ &\quad +a(Y)a_*(\xi)\langle \eta, X\rangle - a(Y)\langle [\xi,\eta], X\rangle \\ &\quad -\langle L_{L_\xi X}\eta, Y\rangle. \end{aligned}$$

The lemma follows immediately.

Lemma 3.4.

$$
\begin{aligned}
&([e_1,e_2],e_3)_- + c.p. \\
&\quad = T(e_1,e_2,e_3) \\
&\qquad + [\{a(X_3)(e_1,e_2)_- + 2a_*(\xi_3)(e_1,e_2)_- \\
&\qquad\quad - \langle[\xi_1,\xi_2],X_3\rangle\} + c.p.].
\end{aligned}
\tag{13}
$$

Proof. By definition,

$$([e_1,e_2],e_3)_- + ([e_1,e_2],e_3)_+ = \langle[e_1,e_2]^*, X_3\rangle,$$

where $[e_1,e_2]^*$ refers to the component of $[e_1,e_2]$ in $\Gamma(A^*)$.

It thus follows that

$$
\begin{aligned}
&\{([e_1,e_2],e_3)_- + c.p.\} + 3T(e_1,e_2,e_3) \\
&\quad = \langle[e_1,e_2]^*, X_3\rangle + c.p. \\
&\quad = \langle[\xi_1,\xi_2] + L_{X_1}\xi_2 - L_{X_2}\xi_1 + d(e_1,e_2)_-,\quad X_3\rangle + c.p. \\
&\quad = \{\langle[\xi_1,\xi_2],X_3\rangle + a(X_1)\langle\xi_2,X_3\rangle - \langle\xi_2,[X_1,X_3]\rangle \\
&\qquad\quad + a(X_3)(e_1,e_2)_- - a(X_2)\langle\xi_1,X_3\rangle + \langle\xi_1,[X_2,X_3]\rangle\} + c.p. \\
&\quad = \{\langle[\xi_1,\xi_2],X_3\rangle + 2\langle[X_1,X_2],\xi_3\rangle + 3a(X_3)(e_1,e_2)_-\} + c.p. \\
&\quad = 4T(e_1,e_2,e_3) + [\{a(X_3)(e_1,e_2)_- + 2a_*(\xi_3)(e_1,e_2)_- \\
&\qquad\qquad\qquad - \langle[\xi_1,\xi_2],X_3\rangle\} + c.p.],
\end{aligned}
$$

where the second from the last step follows essentially from reorganizing cyclic permutation terms, and the last step uses Equation (10). Equation (13) thus follows immediately. q.e.d.

Proof of Theorem 3.1. We denote by I_1 and I_2 the components of $J(e_1,e_2,e_3)$ on $\Gamma(A^*)$ and $\Gamma(A)$ respectively. Thus, by definition,

$$
\begin{aligned}
I_1 \quad = \quad & \{[[\xi_1,\xi_2],\xi_3] + [L_{X_1}\xi_2 - L_{X_2}\xi_1, \xi_3] + [d(e_1,e_2)_-, \xi_3] \\
& + L_{[X_1,X_2] + L_{\xi_1}X_2 - L_{\xi_2}X_1 - d_*(e_1,e_2)_-}\xi_3 \\
& + L_{X_3}L_{X_2}\xi_1 - L_{X_3}L_{X_1}\xi_2 - L_{X_3}[\xi_1,\xi_2] \\
& - d[a(X_3)(e_1,e_2)_-] + d([e_1,e_2],e_3)_-\} + c.p..
\end{aligned}
$$

By using the Jacobi identity: $[[\xi_1,\xi_2],\xi_3] + c.p. = 0$ and the relation: $L_{[X_1,X_2]} = [L_{X_1}, L_{X_2}]$, we can write

$$
\begin{aligned}
I_1 \quad = \quad & \{[L_{X_1}\xi_2 - L_{X_2}\xi_1, \xi_3] + L_{L_{\xi_1}X_2 - L_{\xi_2}X_1}\xi_3 \\
& - L_{d_*(e_1,e_2)_-}\xi_3 + [d(e_1,e_2)_-,\ \xi_3] - L_{X_3}[\xi_1,\xi_2] \\
& + d([e_1,e_2],e_3)_- - d(a(X_3)(e_1,e_2)_-)\} + c.p..
\end{aligned}
\tag{14}
$$

Now,

$$\begin{aligned} L_{X_3}[\xi_1,\xi_2] &= (di_{X_3}+i_{X_3}d)[\xi_1,\xi_2] \\ &= d\langle X_3,[\xi_1,\xi_2]\rangle + i_{X_3}L_{\xi_1}d\xi_2 - i_{X_3}L_{\xi_2}d\xi_1 \\ &\quad +i_{X_3}(d[\xi_1,\xi_2]-L_{\xi_1}d\xi_2+L_{\xi_2}d\xi_1). \end{aligned}$$

Applying Lemma 3.3 twice, we have

$$\begin{aligned} &L_{X_3}[\xi_1,\xi_2]+c.p. \\ &\quad=\{[L_{X_1}\xi_2-L_{X_2}\xi_1,\xi_3]+L_{L_{\xi_1}X_2-L_{\xi_2}X_1}\xi_3+2[d(e_1,e_2)_-,\ \xi_3] \\ &\qquad +2d(a_*(\xi_3)(e_1,e_2)_-)-d\langle X_3,[\xi_1,\xi_2]\rangle \\ &\qquad +i_{X_3}(d[\xi_1,\xi_2]-L_{\xi_1}d\xi_2+L_{\xi_2}d\xi_1)\}+c.p. \end{aligned} \tag{15}$$

Substituting Equation (16) into Equation (15) yields

$$\begin{aligned} I_1 =&\{d[([e_1,e_2],e_3)_- - a(X_3)(e_1,e_2)_- - 2a_*(\xi_3)(e_1,e_2)_- \\ &+\langle[\xi_1,\xi_2],X_3\rangle]-K_1-K_2\}+c.p., \end{aligned}$$

where

$$K_1=i_{X_3}(d[\xi_1,\xi_2]-L_{\xi_1}d\xi_2+L_{\xi_2}d\xi_1)$$

and

$$K_2=L_{d_*(e_1,e_2)_-}\xi_3+[d(e_1,e_2)_-,\ \xi_3].$$

It follows from Lemma 3.4 that

$$I_1=dT(e_1,e_2,e_3)-\{K_1+K_2+c.p.\}.$$

A similar formula for I_2 can be obtained in a similar way. The conclusion follows immediately.

4. Proof of Theorem 2.5

This section is devoted to the proof of Theorem 2.5. Throughout the section, we assume that (A, A^*) is a Lie bialgebroid and $E = A \oplus A^*$ as in Theorem 2.5. We also let $\mathcal{D} : C^\infty(P) \longrightarrow \Gamma(E)$ and $\rho : E \longrightarrow TP$ be defined as in Section 2. To prove Theorem 2.5, it suffices to verify all the five identities in Definition 2.1. First, Equation (i) follows directly from Theorem 3.1 and properties of Lie bialgebroids. Equation (iv) is equivalent to saying that aa_*^* is skew symmetric, which is again a property of a Lie bialgebroid (see Proposition 3.6 in [22]). Below, we shall split the rest of the proof into several propositions.

Proposition 4.1. *For any $f \in C^\infty(P)$ and $e_1, e_2 \in \Gamma(E)$, we have*

$$[e_1, fe_2] = f[e_1, e_2] + (\rho(e_1)f)e_2 - (e_1, e_2)_+\mathcal{D}f. \tag{16}$$

Proof. Suppose that $e_1 = X_1 + \xi_1$ and $e_2 = X_2 + \xi_2$. Then, we have

$$[e_1, fe_2] = [X_1, fX_2] + [X_1, f\xi_2] + [\xi_1, fX_2] + [\xi_1, f\xi_2],$$

where

$$\begin{aligned}
&[X_1, fX_2] = f[X_1, X_2] + (a(X_1)f)X_2;\\
&[\xi_1, f\xi_2] = f[\xi_1, \xi_2] + ((a_*\xi_1)f)\xi_2;\\
&[X_1, f\xi_2] = f[X_1, \xi_2] + ((aX_1)f)\xi_2 - \frac{1}{2}\langle X_1, \xi_2\rangle\mathcal{D}f;\\
&[\xi_1, fX_2] = f[\xi_1, X_2] + ((a_*\xi_1)f)X_2 - \frac{1}{2}\langle X_2, \xi_1\rangle\mathcal{D}f.
\end{aligned}$$

The conclusion follows from adding up all the equations above.

Proposition 4.2. *For any $e_1, e_2 \in \Gamma(E)$, we have*

$$\rho[e_1, e_2] = [\rho e_1, \rho e_2].$$

We need a lemma first.

Lemma 4.3. *If (A, A^*) is a Lie bialgebroid with anchors (a, a_*), then for any $X \in \Gamma(A)$ and $\xi \in \Gamma(A^*)$,*

$$[a(X), a_*(\xi)] = a_*(L_X\xi) - a(L_\xi X) + aa_*^*d_0\langle\xi, X\rangle.$$

Proof. For any $f \in C^\infty(P)$,

$$\begin{aligned}
&(aa_*^*d_0\langle\xi, X\rangle)f\\
&\quad = \langle d_*\langle\xi, X\rangle, df\rangle\\
&\quad = L_{df}\langle\xi, X\rangle\\
&\quad = \langle L_{df}\xi, X\rangle + \langle\xi, L_{df}X\rangle\\
&\quad = -\langle L_\xi df, X\rangle + \langle\xi, [X, d_*f]\rangle\\
&\quad = -a_*(\xi)\langle df, X\rangle + \langle df, L_\xi X\rangle + a(X)a_*(\xi)f - \langle L_X\xi, d_*f\rangle\\
&\quad = [a(X), a_*(\xi)]f - a_*(L_X\xi)f + a(L_\xi X)f,
\end{aligned}$$

where in the fourth equality we have used the fact that $L_{df}X = [X, d_*f]$, a property of a general Lie bialgebroid (see Proposition 3.4 of [22]).

q.e.d.

Proof of Proposition 4.2. Let $e_1 = X_1 + \xi_1$ and $e_2 = X_2 + \xi_2$.

$$\begin{aligned}
\rho[e_1, e_2] &= a\{[X_1, X_2] + L_{\xi_1}X_2 - L_{\xi_2}X_1 - d_*(e_1, e_2)_-\} \\
&\quad + a_*\{[\xi_1, \xi_2] + L_{X_1}\xi_2 - L_{X_2}\xi_1 + d(e_1, e_2)_-\} \\
&= a[X_1, X_2] + a(L_{\xi_1}X_2) - a(L_{\xi_2}X_1) \\
&\quad - \frac{1}{2}aa_*^*d_0(\langle \xi_1, X_2\rangle - \langle \xi_2, X_1\rangle) \\
&\quad + a_*[\xi_1, \xi_2] + a_*(L_{X_1}\xi_2) - a_*(L_{X_2}\xi_1) \\
&\quad + \frac{1}{2}a_*a^*d_0(\langle \xi_1, X_2\rangle - \langle \xi_2, X_1\rangle) \\
&= a[X_1, X_2] + [a(L_{\xi_1}X_2) - a_*(L_{X_2}\xi_1) - aa_*^*d_0\langle \xi_1, X_2\rangle] \\
&\quad - [a(L_{\xi_2}X_1) - a_*(L_{X_1}\xi_2) - aa_*^*d_0\langle \xi_2, X_1\rangle] + a_*[\xi_1, \xi_2] \\
&= [aX_1, aX_2] + [a_*\xi_1, a_*\xi_2] + [aX_1, a_*\xi_2] + [a_*\xi_1, aX_2] \\
&= [\rho(e_1), \rho(e_2)],
\end{aligned}$$

where in the third equality we have used the skew-symmetry of the operator aa_*^*, and the second from the last follows from Lemma 4.3.

Proposition 4.4. *For any $e, h_1, h_2 \in \Gamma(E)$, we have*

$$\begin{aligned}
\rho(e)(h_1, h_2)_+ =& ([e, h_1] + \mathcal{D}(e, h_1)_+,\ h_2)_+ \\
&+ (h_1,\ [e, h_2] + \mathcal{D}(e, h_2)_+)_+.
\end{aligned} \tag{17}$$

Proof. According to Equation (11),

$$([e, h_1], h_2)_+ = T(e, h_1, h_2) + \frac{1}{2}\rho(e)(h_1, h_2)_+ - \frac{1}{2}\rho(h_1)(e, h_2)_+$$

and

$$(h_1, [e, h_2])_+ = T(e, h_2, h_1) + \frac{1}{2}\rho(e)(h_2, h_1)_+ - \frac{1}{2}\rho(h_2)(e, h_1)_+.$$

By adding these two equations, we obtain Equation (17) immediately since $T(e, h_1, h_2)$ is skew-symmetric with respect to h_1 and h_2.

5. Proof of Theorem 2.6

This section is devoted to the proof of Theorem 2.6. We denote sections of L_1 by letters X, Y, and sections of L_2 by ξ, η, etc.. For any $X \in \Gamma(L_1)$ and $\xi \in \Gamma(L_2)$, we define their pairing by

$$\langle \xi, X\rangle = 2(\xi, X). \tag{18}$$

Since $(\cdot,\cdot)$ is nondegenerate, L_2 can be considered as the dual bundle of L_1 under this pairing. Moreover, the symmetric bilinear form $(\cdot,\cdot)_+$ on E defined by Equation (6) coincides with the original one.

By Proposition 2.3, both L_1 and L_2 are Lie algebroids, and their anchors are given by $a = \rho|_{L_1}$ and $a_* = \rho|_{L_2}$ respectively. We shall use $d : \Gamma(\wedge^* L_2) \longrightarrow \Gamma(\wedge^{*+1} L_2)$ and $d_* : \Gamma(\wedge^* L_1) \longrightarrow \Gamma(\wedge^{*+1} L_1)$ to denote their induced de-Rham differentials as usual.

Equation (v) in Definition 2.1 implies immediately that the bracket between $X \in \Gamma(L_1)$ and $\xi \in \Gamma(L_2)$ is given by

$$[X,\xi] = (-L_\xi X + \frac{1}{2} d_*\langle \xi, X\rangle) + (L_X\xi - \frac{1}{2} d\langle \xi, X\rangle). \tag{19}$$

Thus we have

Proposition 5.1. *Under the decomposition $E = L_1 \oplus L_2$, for sections $e_i \in \Gamma(E)$, $i = 1, 2$ if we write $e_i = X_i + \xi_i$, then the bracket $[e_1, e_2]$ is given by Equation (7).*

Before proving Theorem 2.6, we need the following lemma.

Lemma 5.2. *Under the assumption of Theorem 2.6 we have*

$$L_{d_* f}\xi = -[df, \xi],$$
$$L_{df} X = -[d_* f, X],$$

for any $f \in C^\infty(P)$, $X \in \Gamma(L_1)$ and $\xi \in \Gamma(L_2)$.

Proof. Clearly, Equation (iv) in Definition 2.1 yields that $a{\circ}d_* = -a_*{\circ}d$. Therefore,

$$\begin{aligned}
[a_*\xi, aX] &= [\rho\xi, \rho X] \\
&= \rho[\xi, X] \\
&= \rho(L_\xi X - \frac{1}{2} d_*\langle \xi, X\rangle - L_X\xi + \frac{1}{2} d\langle \xi, X\rangle) \qquad (20)\\
&= a(L_\xi X - \frac{1}{2} d_*\langle \xi, X\rangle) + a_*(-L_X\xi + \frac{1}{2} d\langle \xi, X\rangle) \\
&= a(L_\xi X) - a_*(L_X\xi) + (a_* d)\langle \xi, X\rangle.
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
((a_*d)\langle\xi,X\rangle)f &= (a(d_*f))\langle\xi,X\rangle\\
&= \langle L_{d_*f}\xi,X\rangle+\langle\xi,[d_*f,X]\rangle\\
&= \langle[\xi,df],X\rangle-\langle\xi,L_Xd_*f\rangle+\langle L_{d_*f}\xi+[df,\xi],X\rangle\\
&= a_*(\xi)a(X)f-\langle df,L_\xi X\rangle-a(X)a_*(\xi)f\\
&\quad +\langle L_X\xi,d_*f\rangle+\langle L_{d_*f}\xi+[df,\xi],X\rangle\\
&= [a_*(\xi),a(X)]f-a(L_\xi X)f+a_*(L_X\xi)f\\
&\quad +\langle L_{d_*f}\xi+[df,\xi],X\rangle.
\end{aligned}
\tag{21}
$$

Comparing Equation (20) with (22), we obtain

$$\langle L_{d_*f}\xi+[df,\xi],X\rangle=0.$$

Therefore, $L_{d_*f}\xi=-[df,\xi]$. The other equation can be proved similarly. q.e.d.

Proof of Theorem 2.6. It follows from Theorem 3.1 that $J_1+J_2+c.p.=0$, for any e_1,e_2 and $e_3\in\Gamma(E)$. Using Lemma 5.2, we have $J_1+c.p.=0$. In particular, if we take $e_1=X_1,\ e_2=X_2$ and $e_3=\xi_3$, we obtain that $i_{\xi_3}(d_*[X_1,X_2]-L_{X_1}d_*X_2+L_{X_2}d_*X_1)=0$, which implies the compatibility condition between A and A^*. q.e.d.

6. Hamiltonian operators

Throughout this section, we will assume that (A,A^*) is a Lie bialgebroid. Suppose that $H:A^*\longrightarrow A$ is a bundle map. We denote by A_H the graph of H, considered as a subbundle of $E=A\oplus A^*$. I.e., $A_H=\{H\xi+\xi|\xi\in A^*\}$.

Theorem 6.1. *A_H is a Dirac subbundle iff H is skew-symmetric and satisfies the following Maurer-Cartan type equation:*

$$d_*H+\frac{1}{2}[H,H]=0 \tag{22}$$

where H is considered as a section of $\wedge^2A$.

In the sequel, we shall use the same symbol to denote a section of $\wedge^2A$ and its induced bundle map if no confusion is caused.

Proof. It is easy to see that A_H is isotropic iff H is skew-symmetric. For any $\xi,\eta\in\Gamma(A^*)$, let

$$[\xi,\eta]_H=L_{H\xi}\eta-L_{H\eta}\xi+d\langle\xi,H\eta\rangle. \tag{23}$$

From

$$[H\xi, \eta] = L_{H\xi}\eta - L_\eta H\xi - \frac{1}{2}(d - d_*)\langle \eta, H\xi\rangle \text{ and}$$
$$[\xi, H\eta] = L_\xi H\eta - L_{H\eta}\xi + \frac{1}{2}(d - d_*)\langle \xi, H\eta\rangle,$$

it follows that

$$[H\xi, \eta] + [\xi, H\eta] = [\xi, \eta]_H + L_\xi H\eta - L_\eta H\xi + d_*\langle \eta, H\xi\rangle.$$

On the other hand, we have the following formula (see [13]):

$$[H\xi, H\eta] = H[\xi, \eta]_H - \frac{1}{2}[H, H](\xi, \eta). \tag{24}$$

Therefore,

$$\begin{aligned}
[H\xi + \xi, H\eta + \eta] &= [H\xi, H\eta] + [\xi, H\eta] + [H\xi, \eta] + [\xi, \eta] \\
&= \Big(L_\xi H\eta - L_\eta H\xi + d_*\langle \eta, H\xi\rangle \\
&\qquad + H[\xi, \eta]_H - \frac{1}{2}[H, H](\xi, \eta)\Big) \\
&\quad + ([\xi, \eta] + [\xi, \eta]_H),
\end{aligned}$$

so that A_H is integrable iff for any $\xi, \eta \in \Gamma(A^*)$

$$H[\xi, \eta] = L_\xi H\eta - L_\eta H\xi + d_*\langle \eta, H\xi\rangle - \frac{1}{2}[H, H](\xi, \eta). \tag{25}$$

On the other hand,

$$\begin{aligned}
(d_*H)(\xi, \eta, \zeta) &= a_*(\xi)\langle \eta, H\zeta\rangle - a_*(\eta)\langle \xi, H\zeta\rangle + a_*(\zeta)\langle \xi, H\eta\rangle \\
&\quad - \langle [\xi, \eta], H\zeta\rangle + \langle [\xi, \zeta], H\eta\rangle - \langle [\eta, \zeta], H\xi\rangle \\
&= \langle H[\xi, \eta] + L_\eta H\xi - L_\xi H\eta + d_*\langle \xi, H\eta\rangle, \zeta\rangle.
\end{aligned}$$

Hence,

$$(d_*H)(\xi, \eta) = H[\xi, \eta] + L_\eta H\xi - L_\xi H\eta - d_*\langle \eta, H\xi\rangle. \tag{26}$$

This implies that Equation (25) is equivalent to

$$(d_*H)(\xi, \eta) + \frac{1}{2}[H, H](\xi, \eta) = 0,$$

or (22).

Remark (1) Because of the symmetric role of A and A^*, we have the following assertion: the graph $A_I = \{X + IX | X \in \Gamma(A)\}$ of a bundle map $I : A \longrightarrow A^*$ defines a Dirac subbundle iff I is skew-symmetric and satisfies the following Maurer-Cartan type equation:

$$dI + \frac{1}{2}[I, I] = 0. \tag{27}$$

(2) For the canonical Lie bialgebroid (TM, T^*M) where M is equipped with the zero Poisson structure, Equation (22) becomes $[H, H] = 0$, which is the defining equation for a Poisson structure. On the other hand, if we exchange TM and T^*M, and consider the Lie bialgebroid (T^*M, TM), the bracket term drops out of Equation (22), whose solutions correspond to a presymplectic structures. Encompassing these two cases into a general framework was indeed the main motivation for Courant [4] to define and study Dirac structures.

(3) The Maurer-Cartan equation is a kind of integrability equation. It is also basic in deformation theory, where it may live on a variety of differential graded Lie algebras. It would be interesting to place the occurrence of this equation in our theory in a more general context.

Definition 6.2. Given a Lie bialgebroid (A, A^*), a section $H \in \Gamma(\wedge^2 A)$ is called a *hamiltonian operator* if A_H defines a Dirac structure. H is called a *strong hamiltonian operator* if $A_{\lambda H}$ are Dirac subbundles for all $\lambda \in \mathbb{R}$.

Corollary 6.3. *For a Lie bialgebroid (A, A^*), $H \in \Gamma(\wedge^2 A)$ is a hamiltonian operator if Equation (22) holds. It is a strong hamiltonian operator if $d_* H = [H, H] = 0$.*

For a hamiltonian operator H, A_H is a Dirac subbundle which is transversal to A in $A \oplus A^*$. Therefore, (A, A_H) is a Lie bialgebroid according to Theorem 2.6. In fact, A_H is isomorphic to A^*, as a vector bundle, with the anchor and bracket of its Lie algebroid structure given respectively by $\hat{a}_* = a_* + a \circ H$ and $\widehat{[\xi, \eta]} = [\xi, \eta] + [\xi, \eta]_H$, for all $\xi, \eta \in \Gamma(A^*)$. In particular, if H is a strong hamiltonian operator, one obtains a one-parameter family of Lie bialgebroids transversal to A, which can be considered as a deformation of the Lie bialgebroid (A, A^*).

Conversely, any Dirac subbundle transversal to A corresponds to a hamiltonian operator in an obvious way. For example, consider the standard Lie bialgebra $(\mathfrak{k}, \mathfrak{b})$ arising from the Iwasawa decomposition of

$\mathfrak{k}^{\mathbb{C}}$ [19], where $\mathfrak{k}$ is a compact semi-simple Lie algebra and $\mathfrak{b}$ its corresponding dual Lie algebra. Then any real form of $\mathfrak{k}^{\mathbb{C}}$ which is transversal to $\mathfrak{b}$ will correspond to a hamiltonian operator (see [16] for a complete list of such real forms for simple Lie algebras). It is straightforward to check that such a hamiltonian operator is not strong. On the other hand, for the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{k}$, every element in $\wedge^2 h$ gives rise to a strong hamiltonian operator $H : \mathfrak{k}^* \longrightarrow \mathfrak{k}$.

This gives rise to a deformation of the standard Lie bialgebra $(\mathfrak{k}, \mathfrak{b})$ (see [15]).

These examples can be generalized to any gauge algebroid associated to a principal K-bundle.

Below, we will look at hamiltonian operators in two special cases, each of which corresponds to some familiar objects.

Example 6.4. Let P be a Poisson manifold with Poisson tensor π. Let (TP, T^*P) be the canonical Lie bialgebroid associated to the Poisson manifold P and $E = TP \oplus T^*P$ equipped with the induced Courant algebroid structure. It is easy to see that a bivector field H is a hamiltonian operator iff $H + \pi$ is a Poisson tensor. H is a strong hamiltonian operator iff H is a Poisson tensor Schouten-commuting with π.

Example 6.5. Similarly, we may switch TP and T^*P, and consider the Lie bialgebroid (T^*P, TP) associated to a Poisson manifold P with Poisson structure π. Let $E = T^*P \oplus TP$ be equipped with its Courant algebroid structure. In this case, a hamiltonian operator corresponds to a two-form $\omega \in \Omega^2(P)$ satisfying $d\omega + \frac{1}{2}[\omega, \omega]_\pi = 0$. Here, $[\cdot, \cdot]_\pi$ refers to the Schouten bracket of differential forms on P induced by the Poisson structure π. Given a hamiltonian operator ω, its graph A_ω defines a Dirac subbundle transversal to T^*P, the first component of E also being considered as a Dirac subbundle. Therefore, they form a Lie bialgebroid. Their induced Poisson structure on the base space can be easily checked to be given by $-2(\pi^\# + N\pi^\#)$, where $N : TP \longrightarrow TP$ is the composition $\pi^\# \circ \omega^b$, and $\omega^b : TP \longrightarrow T^*P$ is the bundle map induced by the two-form ω. If ω is a strong hamiltonian operator, then $N\pi^\#$ defines a Poisson structure compatible with π. In fact, in this case, (π, N) is a Poisson-Nijenhuis structure in the sense of [13].

We note that Vaisman [31] has studied 2-forms on Poisson manifolds satisfying the condition $[\omega, \omega]_\pi = 0$. Such forms, called *complementary* to the Poisson structure, also give rise to new Lie algebroid structures on TM.

To end this section, we describe a example of Lie bialgebroids, where both the algebroid and its dual arise from hamiltonian operators.

Proposition 6.6. *Let U and V be Poisson tensors over a manifold M and denote by T^*M_U and T^*M_V their associated canonical cotangent Lie algebroids on T^*M. Assume that $U-V$ is nondegenerate. Then (T^*M_U, T^*M_V) is a Lie bialgebroid, where their pairing is given by $(\xi,\eta)=(U-V)(\xi,\eta)$ for any $\xi\in T^*M_U$ and $\eta\in T^*M_V$. Furthermore their induced Poisson tensor on the base space M is given by $-2U(U-V)^{-1}V$.*

Proof. Let $E=TM\oplus T^*M$ be equipped with the usual Courant bracket. Since both U and V are Poisson tensors, their graphs A_U and A_V are Dirac subbundles, and they are transversal since $U-V$ is nondegenerate. Therefore, (A_U, A_V) is a Lie bialgebroid, where their pairing is given by

$$\langle\!\langle U\eta+\eta, V\xi+\xi\rangle\!\rangle = \frac{1}{2}\langle \xi, (U-V)\eta\rangle. \tag{28}$$

On the other hand, as Lie algebroids, A_U and A_V are clearly isomorphic to cotangent Lie algebroids T^*M_U and T^*M_V respectively. Moreover, their anchors $a_U: A_U\longrightarrow TM$ and $a_V: A_V\longrightarrow TM$ are given respectively by $a_U(U\xi+\xi)=U\xi$ and $a_V(V\xi+\xi)=V\xi$.

This proves the first part of the proposition. To calculate their induced Poisson structure on the base M, we need to find out the dual map $a_V^*: T^*M\longrightarrow A_V^*\cong A_U$. For any $\xi\in T^*M$, we assume that $a_V^*(\xi)=U\eta+\eta\in A_U$ via the identification above. For any $\zeta\in T^*M$,

$$\begin{aligned}(a_V^*\xi, V\zeta+\zeta) &= \langle \xi, a_V(V\zeta+\zeta)\rangle\\ &= \langle \xi, V\zeta\rangle.\end{aligned}$$

On the other hand, $(a_V^*\xi, V\zeta+\zeta)=\langle\!\langle U\eta+\eta, V\zeta+\zeta\rangle\!\rangle=\frac{1}{2}\langle\zeta,(U-V)\eta\rangle$. It thus follows that $\eta=-2(U-V)^{-1}V\xi$. Therefore, according to Proposition 3.6 in [22], the induced Poisson structure $a_U\circ a_V^*: T^*M\longrightarrow TM$ is given by $(a_U\circ a_V^*)\xi=-2U(U-V)^{-1}V\xi$. q.e.d.

Replacing V by $-V$ in the proposition above, we obtain the following "composition law" for Poisson structures.

Corollary 6.7. *Let U and V be Poisson tensors over manifold M such that $U+V$ is nondegenerate. Then, $U(U+V)^{-1}V$ also defines a Poisson tensor on M.*

Note that, if U and V are nondegenerate, then $U(U+V)^{-1}V = (U^{-1}+V^{-1})^{-1}$ is the Poisson tensor corresponding to the sum of the symplectic forms for U and V. Since the sum of closed forms is closed, it is obvious in this case that $U(U+V)^{-1}V$ is a Poisson tensor. We do not know such a simple proof of Corollary 6.7 in the general case.

7. Null Dirac structures and Poisson reduction

In this section, we consider another class of Dirac structures related to Poisson reduction and dual pairs of Poisson manifolds.

Proposition 7.1. *Let (A, A^*) be a Lie bialgebroid, and $h \subseteq A$ a subbundle of A. Then $L = h \oplus h^\perp \subseteq E = A \oplus A^*$ is a Dirac structure iff h and $h^\perp$ are, respectively, Lie subalgebroids of A and A^*.*

Proof. Obviously, $L = h \oplus h^\perp$ is a maximal isotropic subbundle of E. If L is a Dirac structure, clearly h and $h^\perp$ are Lie subalgebroids of A and A^* respectively. Conversely, suppose that both h and $h^\perp$ are Lie subalgebroids of A and A^* respectively. To prove that L is a Dirac structure, it suffices to show that $[X, \xi]$ is a section of L for any $X \in \Gamma(h)$ and $\xi \in \Gamma(h^\perp)$. Now

$$[X, \xi] = L_X \xi - L_\xi X.$$

For any section $Y \in \Gamma(h)$,

$$< L_X\xi,\ Y >= a(X) < \xi, Y > - < \xi, [X, Y] >= 0.$$

Therefore, $L_X\xi$ is still a section of $h^\perp$. Similarly, $L_\xi X$ is a section of h. This concludes the proof of the proposition. q.e.d.

It is clear that a subbundle $L \subseteq E$ is of the form $L = h \oplus h^\perp$ iff the minus two-form $(\cdot, \cdot)_-$ on E, as defined by Equation (6), vanishes on L. For this reason, we call a Dirac structure of this form a *null Dirac structure.*

An immediate consequence of Proposition 7.1 is the following:

Corollary 7.2. *Let $(P,\ \pi)$ be a Poisson manifold, and D a subbundle of TP. Let T^*P be equipped with the cotangent Lie algebroid structure so that $(TP,\ T^*P)$ is a Lie bialgebroid. Then $L = D \oplus D^\perp$ is a Dirac structure in $E = TP \oplus T^*P$ iff D is an integrable distribution and the Poisson structure on P descends to a Poisson structure on the* *quotient space P/D* [4] *such that the natural projection is a Poisson map.*

Proof. This follows directly from the following lemma.

Lemma 7.3. *Let D be an integrable distribution on a Poisson manifold P. Then P/D has an induced Poisson structure (in the above general sense) iff $D^{\perp} \subset T^*P$ is a subalgebroid*[5].

Proof. For simplicity, let us assume that P/D is a manifold. The general case will follow from the same principle. It is clear that a function f is constant along leaves of D iff df is a section of $D^{\perp}$. If $D^{\perp}$ is a subalgebroid, then the equation

$$d\{f,\ g\} = [df, dg] \tag{29}$$

implies that $C^{\infty}(P/D)$ is a Poisson algebra.

Conversely, a local one-form fdg is in $D^{\perp}$ iff g is constant along D. The conclusion thus follows from Equation (29) together with the Lie algebroid axiom relating the bracket and anchor.

Remark. Poisson reduction was considered by Marsden and Ratiu in [24]. Lemma 7.3 can be considered as a special case of their theorem when $P = M$ in the Poisson triple $(P,\ M,\ E)$ (see [24]). It would be interesting to interpret their general reduction theorem in terms of Dirac structures as in Corollary 7.2.

The rest of the section is devoted to several examples of Corollary 7.2, which will lead to some familiar results in Poisson geometry.

Recall that, given a Poisson Lie group G and a Poisson manifold M, an action

$$\sigma : G \times M \longrightarrow M$$

is called a Poisson action if σ is a Poisson map. In this case, M is called a Poisson G-space.

Now consider $P = G \times M$ with the product Poisson structure and diagonal G-action. Then P/G is isomorphic to M, and the projection from P to $P/G = M$ becomes the action map σ, which is a Poisson map when P/G is equipped with the given Poisson structure on M. By Corollary 7.2, we obtain a null Dirac structure $L = D \oplus D^{\perp}$ in $TP \oplus T^*P$.

[4] When the quotient space is not a manifold, this means that at each point there is a local neighborhood U such that the Poisson structure on U descends to its quotient.

[5] Such a foliation is also called a cofoliation by Vaisman [30]

Clearly, L is a Lie algebroid over P, which is G-invariant. It would be interesting to explore the relation between this algebroid and the one defined on $(M \times \mathfrak{g}) \oplus T^*M$, which was studied by Lu in [18].

For a Poisson Lie group G with tangential Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$, the Courant algebroid double $E = TG \oplus T^*G$ can be identified, as a vector bundle, with the trivial product $G \times (\mathfrak{g} \oplus \mathfrak{g}^*)$ via left translation. Under such an identification, a left invariant null Dirac structure has the form $L = G \times (h \oplus h^\perp)$, where h is a subalgebra of $\mathfrak{g}$ and $h^\perp$ is a subalgebra of $\mathfrak{g}^*$. Thus, one obtains the following reduction theorem: for a connected closed subgroup H with Lie algebra h, G/H has an induced Poisson structure iff $h^\perp$ is a subalgebra of $\mathfrak{g}^*$.

More generally, let G be a Poisson group, M a Poisson G-space. Suppose that $H \subseteq G$ is a closed subgroup with Lie algebra h. Assume that M/H is a nice manifold such that the projection $p : M \longrightarrow M/H$ is a submersion. Then the H-orbits define an integrable distribution $\mathfrak{h}$ on M. According to Corollary 7.2, the Poisson structure on M descends to M/H iff $\mathfrak{h}^\perp$ is a subalgebroid of the cotangent algebroid T^*M of the Poisson manifold M. On the other hand, we have

Proposition 7.4. *If $h^\perp$ is a subalgebra of $\mathfrak{g}^*$, then $\mathfrak{h}^\perp$ is a subalgebroid of T^*M. Conversely, if the isotropic subalgebra at each point is a subalgebra of h, and in particular if the action is locally free, then that $\mathfrak{h}^\perp$ is a subalgebroid implies that $h^\perp \subseteq \mathfrak{g}^*$ is a subalgebra.*

Proof. It is easy to see that $\mathfrak{h}^\perp \cong \varphi^{-1}(h^\perp)$, where $\varphi : T^*M \longrightarrow \mathfrak{g}^*$ is the momentum mapping for the lifted G-action on T^*M, equipped with the canonical cotangent symplectic structure. According to Proposition 6.1 in [34], $\varphi : T^*M \longrightarrow \mathfrak{g}^*$ is a Lie algebroid morphism. Before continuing, we need

Lemma 7.5. *Let $A \longrightarrow M$ be a Lie algebroid with anchor a, $\mathfrak{g}$ a Lie algebra, and $\varphi : A \longrightarrow \mathfrak{g}$ an algebroid morphism. Suppose that $h \subseteq \mathfrak{g}$ is a subalgebra such that $\varphi^{-1}h \subseteq A$ is a subbundle. Then $\varphi^{-1}h$ is a subalgebroid.*

Conversely, given a subalgebroid $B \subseteq A$, if $\varphi(B|_m)$ is independent of $m \in M$, then it is a subalgebra of $\mathfrak{g}$.

Proof. This follows directly from the following equation (see [21]):

$$\varphi[X, Y] = (aX)(\varphi Y) - (aY)(\varphi X) + [\varphi X, \ \varphi Y]^{\cdot}, \quad \forall X, Y \in \Gamma(A),$$

where φX, φY and $\varphi[X, Y]$ are considered as $\mathfrak{g}$-valued functions on P, and $[\cdot, \cdot]^{\cdot}$ refers to the pointwise bracket. q.e.d.

Now, the first part of Proposition 7.4 is obvious according to the lemma above. For the second part, we only need to note that $\varphi(\mathfrak{h}^{\perp}) = h^{\perp} \cap Im\varphi$, and the assumption that the isotropic subalgebra at each point is a subalgebra of h is equivalent to that $h^{\perp} \subseteq Im\varphi$. This concludes the proof of the Proposition. q.e.d.

From the above discussion, we have the following conclusion: if $h^{\perp}$ is a subalgebra of $\mathfrak{g}^*$, then the Poisson structure on M descends to M/H. This is a well-known reduction theorem of Semenov-Tian-Shansky [29] (see also [33]).

Conversely, if the isotropic subalgebra at each point is a subalgebra of h, and in particular if the action is locally free, the converse also holds.

Another interesting example arises when P is a symplectic manifold with an invertible Poisson tensor π. In this case, $\pi^{\#} : T^*P \longrightarrow TP$ is a Lie algebroid isomorphism. Given a null Dirac structure $L = D \oplus D^{\perp}$, $\bar{D} = \pi^{\#}(D^{\perp})$ is a subalgebroid of TP. It is simple to see that $(\bar{D})^{\perp} = (\pi^{\#})^{-1}(D)$, and is therefore a subalgebroid of T^*P. Thus, $\bar{L} \stackrel{def}{=} \bar{D} \oplus (\bar{D})^{\perp}$ defines another null Dirac structure. It is easily seen that D and $\bar{D}$ are symplectically orthogonal to each another. Thus $P/\bar{D}$ is a Poisson manifold (assume that it is a nice manifold) so that P/D and $P/\bar{D}$ constitute a full dual pair, which is a well-known result of Weinstein [32]. Conversely, it is clear that a full dual pair corresponds to a null Dirac structure.

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